

# ENGINEERING MATHEMATICS

(For ESE & GATE Exam)

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## **IES MASTER PUBLICATION**

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## PREFACE

Science and technology has contributed greatly to the progress of the human race, and thus has made our life quite easier and comfortable. Engineers have played a vital role in the advancements of the technological world. However, with advancements in technology, there come challenges. It is here that the role of Engineering Mathematics comes in play to address the technological challenges posed by the modern era.

Engineering Mathematics is the art of applying mathematics to complicated real-world problems. It is a mix of mathematical theory, practical engineering and scientific computing.

Keeping in mind the importance of Engineering Mathematics skill in modern engineers, in the year 2017, the Union Public Service Commission (UPSC) introduced Engineering Mathematics as a common paper in the syllabus for Engineering Services Examination (ESE), and technical paper for Electrical (EE) stream. It has already been given a weightage of 15% in the Graduate Aptitude Test in Engineering (GATE)

With an objective to develop an ESE or GATE aspirant's numerical abilities and calculation skills, IES Master has come up with this Engineering Mathematics book that brings them face to face with 280 topics under 32 chapters in 8 units, along with previous years questions from GATE (last 27 years), and ESE (last 3 years) and their detailed solutions. Equipped with all this, students can easily decide, how much time to allocate on each chapter based on the number of questions asked in that particular exam.

It is the only book in the market that includes complete theory exactly on the basis of ESE & GATE exams pattern. After each topic, the book includes more than 640 solved examples for concept building & easy learning. To save students time during revision, all the previously asked conceptual questions are marked with '\*' symbol.

Our special thanks to the entire IES Master team for their continuous support in bringing out the book. We strongly believe that this book will help students in their journey to success. Suggestions from students, teachers & educators for further improvement in the book are always welcome.

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# CONTENTS

## UNIT 1 : LINEAR ALGEBRA

<b>1.1</b>	<b>Algebra of Matrices</b> .....	<b>01 – 34</b>
	(i) Definition of Matrix .....	01
	(ii) Types of Matrices .....	01
	(iii) Product of Matrix by a Scalar (or Constant) .....	03
	(iv) Addition and Subtraction of Matrices .....	03
	(v) Multiplication of Matrices .....	04
	(vi) Minors of Matrix .....	05
	(vii) Cofactors of a Matrix .....	05
	(viii) Properties of Determinants .....	06
	(ix) Adjoint of Matrix .....	06
	(x) Inverse of a Matrix .....	07
	(xi) Conjugate of a Matrix .....	07
	(xii) Conjugate Transpose of a Matrix .....	08
	(xiii) Special Types of Matrices .....	08
	<b>Previous Years GATE &amp; ESE Questions</b> .....	<b>14</b>
<b>1.2</b>	<b>Rank of Matrices</b> .....	<b>35 – 54</b>
	(i) Definition of Rank .....	35
	(ii) Elementary Transformations or E-Operation .....	36
	(iii) Equivalent Matrices .....	36
	(iv) Properties of Rank .....	36
	(v) Echelon Form / Triangular Form .....	37
	(vi) Normal Form of a Matrix (Canonical Form) .....	39
	(vii) Elementary Matrices .....	42
	(viii) Computation of the Inverse of Matrix by Elementary Transformation .....	43
	(ix) Row Rank and Column Rank of a Matrix .....	46
	<b>Previous Years GATE &amp; ESE Questions</b> .....	<b>48</b>
<b>1.3</b>	<b>System of Equations</b> .....	<b>55 – 95</b>
	(i) Linear Dependence and Independence of Vectors .....	55
	(ii) System of Linear Equations .....	58

(iii)	Methods of Solving Non-Homogenous System .....	59
(iv)	Methods of Solving Homogenous System .....	66
(v)	LU Decomposition (Factorization) Method .....	70
(vi)	Dolittle's Method .....	70
(vii)	Crout's Method .....	73
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>76</b>

#### **1.4 Eigen Values and Eigen Vectors ..... 96 – 142**

(i)	Definition .....	96
(ii)	Algebraic & Geometric Multiplicity .....	97
(iii)	Properties of Eigen Values & Eigen Vectors .....	97
(iv)	Similar Matrices & Diagonalisation .....	104
(v)	Cayley-Hamilton Theorem .....	105
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>109</b>

#### **1.5 Pivoting & Vector Space ..... 143 – 151**

(i)	Normalisation .....	143
(ii)	ILL-Conditioned System .....	143
(iii)	Diagonally Dominant System .....	143
(iv)	Gauss Elimination Method with Pivoting .....	143
(v)	Gauss-Jordan Method .....	146
(vi)	Vector Space .....	147
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>149</b>

### **UNIT 2 : CALCULUS**

#### **2.1 Functions and their Graphs ..... 152 – 165**

(i)	Functions .....	152
(ii)	Types of Functions .....	152
(iii)	Some Basic Graphs .....	153
(iv)	Graph Transformation .....	159
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>160</b>

#### **2.2 Infinite Series ..... 166 – 179**

(i)	Expansion of Functions .....	166
(ii)	Maclaurin's Theorem (Series) .....	166
(iii)	Taylor's Theorem (Series) .....	166
(iv)	Convergence and Divergence of Infinite Series .....	169
(v)	Methods to Find Convergence of Infinite Series .....	170
(vi)	Power Series .....	171
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>173</b>

<b>2.3</b>	<b>Limits, Continuity and Differentiability .....</b>	<b>180– 222</b>
	(i) Function .....	180
	(ii) Limit of a Function .....	180
	(iii) Theorem on Limits .....	181
	(iv) Indeterminate Forms .....	183
	(v) L-Hospital Rule .....	184
	(vi) Fundamentals of Continuity .....	187
	(vii) Kinds of Discontinuities .....	188
	(viii) Properties of Continuous Functions .....	194
	(ix) Function of Two Variables .....	195
	(x) Limit of a Function of Two Variables .....	195
	(xi) Continuity of Function of Two Variables .....	200
	(xii) Differentiability .....	202
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>207</b>
<b>2.4</b>	<b>Mean Value Theorems .....</b>	<b>223 – 231</b>
	(i) Rolle's Theorem .....	223
	(ii) Lagrange's First Mean Value Theorem .....	224
	(iii) Cauchy's Mean Value Theorem .....	224
	(iv) Bolzano Theorem .....	225
	(v) Intermediate Value Theorem .....	225
	(vi) Darboux's Theorem .....	225
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>227</b>
<b>2.5</b>	<b>Maxima and Minima .....</b>	<b>232 – 261</b>
	(i) Increasing-Decreasing Function .....	232
	(ii) Maxima and Minima of Function of Single Variable .....	233
	(iii) Maxima-Minima of Functions of Two Variables .....	236
	(iv) Sufficient Condition (Lagrange's Conditions) .....	237
	(v) Lagrange's Method of Undetermined Multipliers .....	240
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>245</b>
<b>2.6</b>	<b>Partial Differentiation .....</b>	<b>262 – 285</b>
	(i) Partial Derivatives .....	262
	(ii) Homogeneous Functions .....	267
	(iii) Euler's Theorem on Homogenous Function .....	268
	(iv) Total Differential Coefficient .....	270
	(v) Change of Variables .....	270

(vi) Jacobian .....	274
(vii) Chain Rule of Jacobians .....	277
(viii) Functional Dependence .....	278
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>281</b>

**2.7 Multiple Integral ..... 286 – 348**

(i) Concept of Integration .....	286
(ii) Some Standard Formulae .....	286
(iii) Some Important Integration and their Hints .....	287
(iv) Definite integrals .....	288
(v) Fundamental Properties of Definite Integrals .....	288
(vi) Fundamental Theorem of Integral Calculus .....	290
(vii) Definite integral as the Limit of a Sum .....	290
(viii) Leibnitz Rule of Differentiation Under the Sign of Integration .....	293
(ix) Improper Integral .....	294
(x) Beta and Gamma Functions .....	295
(xi) Properties of Beta and Gamma Functions .....	295
(xii) Double Integrals .....	302
(xiii) Double Integrals in Polar Coordinates .....	303
(xiv) Triple Integrals .....	309
(xv) Change of Order of Integration .....	311
(xvi) Use of Jacobian in Multiple Integrals .....	314
(xvii) Multiple Integral using Change of Variables Concept .....	316
(xviii) Area and Volume in Different Coordinates Systems .....	317
(xix) Arc Length of Curves (Rectification) .....	320
(xx) Intrinsic Equation of a Curve .....	320
(xxi) Volumes of Solids of Revolution .....	323
(xxii) Surfaces of Solids of Revolution .....	324
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>326</b>

**UNIT 3 : VECTOR CALCULUS**

**3.1 Vector & their Basic Properties ..... 349 – 367**

(i) Vector .....	349
(ii) Dot Product .....	350
(iii) Cross Product .....	352
(iv) Scalar Triple Product .....	354

	(v) Vector Triple Product .....	354
	(vi) Direction Cosines & Direction Ratios .....	355
	(vii) Straight Lines in 3D .....	356
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>363</b>
<b>3.2</b>	<b>Gradient, Divergence and Curl .....</b>	<b>368 – 395</b>
	(i) Partial Derivatives of Vectors .....	368
	(ii) Point Functions .....	368
	(iii) Del Operator .....	369
	(iv) The Laplacian Operator .....	369
	(v) Gradient .....	370
	(vi) Directional Derivative (D.D.) .....	370
	(vii) Level Surface .....	371
	(viii) Divergence .....	376
	(ix) Curl (Rotation) .....	377
	(x) Vector Identities .....	378
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>382</b>
<b>3.3</b>	<b>Vector Integration .....</b>	<b>396 – 414</b>
	(i) Line Integral .....	396
	(ii) Surface Integral .....	398
	(iii) Volume Integral .....	399
	(iv) Gauss' Divergence Theorem .....	400
	(v) Stoke's Theorem .....	402
	(vi) Green's Theorem .....	403
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>405</b>

#### UNIT 4 : COMPLEX ANALYSIS

<b>4.1</b>	<b>Complex Number .....</b>	<b>415 – 431</b>
	(i) IOTA .....	415
	(ii) Complex Numbers as Ordered Pairs .....	415
	(iii) Euler Notation .....	416
	(iv) Geometrical Representation .....	416
	(v) Algebraic Properties .....	417
	(vi) Complex Conjugate Numbers .....	418
	(vii) Modulus and Argument .....	419
	(viii) Equation of a Circle in Complex Form .....	422



(ix)	Roots of a Complex Number .....	422
(x)	Complex No. as a Rotating Arrow .....	424
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>425</b>
<b>4.2</b>	<b>Analytic Functions .....</b>	<b>432 – 457</b>
(i)	Neighbourhood of Complex Number $z_0$ .....	432
(ii)	Function of a Complex Variable .....	432
(iii)	Types of Complex Function .....	433
(iv)	Complex Mapping/Transformation .....	433
(v)	Continuity of Complex Function .....	435
(vi)	Differentiability of Complex Function .....	435
(vii)	Analytic Functions/Regular Function/Holomorphic Function .....	436
(viii)	Cauchy-Riemann Equations .....	437
(ix)	Polar Form of Cauchy-Riemann Equations .....	439
(x)	Orthogonal System .....	441
(xi)	Construction of an Analytic Function .....	445
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>447</b>
<b>4.3</b>	<b>Complex Integration .....</b>	<b>458 – 491</b>
(i)	Contour .....	458
(ii)	Elementary Properties of Complex Integrals .....	459
(iii)	Cauchy Integral Theorem .....	462
(iv)	Cauchy's Integral Formula .....	463
(v)	Taylor Series of Complex Function .....	468
(vi)	Laurent's Series .....	468
(vii)	Zeroes of an Analytic Function .....	469
(viii)	Singularities of an Analytic Function .....	470
(ix)	Types of Isolated Singular Points .....	470
(x)	Residue .....	471
(xi)	Cauchy-Residue Theorem .....	472
(xii)	Methods of Evaluating Residues .....	472
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>475</b>

## UNIT 5 : DIFFERENTIAL EQUATIONS

<b>5.1</b>	<b>Ordinary Differential Equations .....</b>	<b>492 – 523</b>
(i)	Definition of Differential Equations .....	492
(ii)	Order and Degree of Ordinary Differential Equation .....	492

(iii)	Non-Linear Differential Equation .....	493
(iv)	Solution of Differential Equations .....	493
(v)	Formation of Differential Equation .....	494
(vi)	Wronskian .....	495
(vii)	Linearly Dependent (LD) & Linearly Independent (LI) Solutions .....	495
(viii)	Methods of Solving Differential Equation .....	496
(ix)	Differential Equations of First Order and First Degree .....	496
(x)	Exact Differential Equation .....	498
(xi)	Equation Reducible to Exact Form .....	499
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>504</b>
<b>5.2</b>	<b>Linear Differential Equations of Higher Order with Constant Coefficients .....</b>	<b>524 – 569</b>
(i)	Complementary Function (C.F.) .....	524
(ii)	Rules for Finding the Complementary Function .....	524
(iii)	Particular Integral (P.I.) .....	527
(iv)	Methods of Evaluating (P.I.) .....	527
(v)	Cauchy's Homogeneous Linear Differential Equation .....	535
(vi)	Legendre's Homogeneous Linear Differential Equations .....	535
(vii)	Simultaneous Linear Differential Equation .....	539
(viii)	Variation of Parameters .....	539
(ix)	Differential Equations of First Order and Higher Degree .....	540
(x)	Clairaut's Equation .....	543
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>545</b>
<b>5.3</b>	<b>Partial Differential Equations .....</b>	<b>570 – 592</b>
(i)	Order and Degree of Partial Differential Equation .....	570
(ii)	Linear P.D. Equation .....	570
(iii)	Lagrange's Method of Solving Linear .....	571
(iv)	Homogenous Linear P.D.E. with Constant Coefficient .....	574
(v)	Non Homogeneous Linear P.D.E. with Constant Coefficient .....	574
(vi)	Classification of 2nd Order Linear P.D.E. in Two Variables .....	582
(vii)	One Dimensional Wave Equation .....	583
(viii)	One Dimensional Heat Equation .....	584
(ix)	Two Dimensional Heat Equation .....	585
(x)	Laplace Equation .....	586

(xi) Clairaut Ordinary Differential Equation .....	587
(xii) Clairaut Partial Differential Equation .....	587
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>588</b>

## UNIT 6 : NUMERICAL METHODS

<b>6.1 Numerical Solutions of Linear Equations .....</b>	<b>593 – 602</b>
(i) Gauss-Jacobi Method .....	593
(ii) Gauss-Seidel Method of Iteration .....	596
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>601</b>
<b>6.2 Numerical Solution of Algebraic and Transcendental Equations .....</b>	<b>603 – 626</b>
(i) Graphical Method .....	603
(ii) Bisection Method .....	604
(iii) Regula-Falsi Method .....	608
(iv) The Secant Method .....	610
(v) Newton-Raphson Method .....	611
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>615</b>
<b>6.3 Interpolation .....</b>	<b>627 – 635</b>
(i) Interpolation and Extrapolation .....	627
(ii) Various Methods of Interpolation .....	627
(iii) Notation of Finite Difference Calculus .....	627
(iv) Newton-Gregory Forward Interpolation Formula for Equal Intervals.....	628
(v) Newton-Gregory Backward Interpolation Formula for Equal Intervals .....	629
(vi) Newton's Divided Difference Interpolation Formula for Unequal Intervals .....	631
(vii) Lagrange's Interpolation Formula for Unequal Intervals .....	632
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>634</b>
<b>6.4 Numerical Integration (Quadrature) .....</b>	<b>636 – 651</b>
(i) General Quadrature Formula .....	636
(ii) The Trapezoidal Rule .....	636
(iii) Simpson's One-Third Rule .....	637
(iv) Simpsons 3/8 <sup>th</sup> Rule .....	638
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>644</b>
<b>6.5 Numerical Solution of a Differential Equation .....</b>	<b>652 – 668</b>
(i) Picard's Method of Successive Approximation .....	652

(ii) Euler's Method/Forward Euler Method/Explicit Euler Method .....	655
(iii) Backward Euler's Method/Implicit Euler Method .....	655
(iv) Runge-Kutta Methods .....	658
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>664</b>

### UNIT 7 : TRANSFORM THEORY

<b>7.1 Laplace Transform .....</b>	<b>669 – 696</b>
(i) Definition of Laplace Transform .....	669
(ii) Properties of Laplace Transform .....	669
(iii) Laplace Transform of Periodic Function .....	670
(iv) Types of Laplace Transform .....	670
(v) Evaluation of Integrals with the help of Laplace Transform .....	672
(vi) Inverse Laplace Transform .....	672
(vii) Properties of Inverse Laplace Transform .....	673
(viii) Types of Inverse Laplace Transform .....	673
(ix) Convolution of Two Functions (Faltung of Two Functions) .....	674
(x) Unit Step Function (Heaviside's Unit Step Function) .....	674
(xi) Unit Impulse Function (Dirac Delta Function) .....	674
(xii) Existence Theorem for Laplace Transform .....	676
(xiii) Initial Value Theorem .....	676
(xiv) Final Value Theorem .....	677
(xv) Application of Laplace Transform in Differential Equation .....	677
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>679</b>
<b>7.2 Fourier Series .....</b>	<b>697 – 709</b>
(i) Periodic Function .....	697
(ii) Fourier Series : (Main Definition) .....	698
(iii) Euler's Formula of Fourier Series .....	698
(iv) Dirichlet's Conditions .....	700
(v) Fourier Expansion of Even and Odd Function .....	701
(vi) Half Range Fourier Series .....	703
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>705</b>
<b>7.3 Fourier and Z-Transforms .....</b>	<b>710 – 720</b>
(i) Integral Transform .....	710
(ii) Fourier Integral Theorem .....	710

(iii) Fourier Transforms .....	711
(iv) Z-Transform (Definition) .....	715
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>719</b>

## UNIT 8 : PROBABILITY & STATISTICS

<b>8.1 Probability .....</b>	<b>721 – 774</b>
(i) Some Basic Concepts .....	721
(ii) Representation of Various Events Based on Algebraic Operations .....	724
(iii) Probability (Definition) .....	725
(iv) Addition Theorem of Probability .....	729
(v) Odds in Favour and Odds Against .....	732
(vi) Independent Events .....	733
(vii) Conditional Probability .....	737
(viii) Binomial Theorem on Probability .....	740
(ix) Total Probability Theorem .....	741
(x) Baye's Theorem (Inverse Probability Theorem) .....	742
<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>746</b>
<b>8.2 Statistics - I (Probability Distributions) .....</b>	<b>775 – 826</b>
(i) Random Experiment .....	775
(ii) Discrete Random Variable .....	775
(iii) The Bernoulli Random Variable .....	777
(iv) The Binomial Random Variable .....	777
(v) The Geometric Random Variable .....	779
(vi) The Poisson Random Variable .....	779
(vii) Continuous Random Variable .....	780
(viii) Uniform Random Variable .....	782
(ix) Exponential Random Variable .....	783
(x) Gamma Random Variable .....	783
(xi) Normal Random Variable/Normal Distribution/Gaussian Distribution .....	783
(xii) Standard Normal Distribution .....	784
(xiii) Skewness in Normal Curve .....	785
(xiv) Expectation of Random Variable .....	785
(xv) Variance .....	787
(xiv) Standard Deviation .....	787

(xvii)	Covariance .....	787
(xviii)	Cumulative Distribution Function (CDF) for Discrete Random Variable .....	788
(xix)	Cumulative Distribution Function for Continuous Random Variable .....	790
(xx)	Mean .....	790
(xxi)	Median .....	790
(xxii)	Mode .....	791
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>792</b>
<b>8.3</b>	<b>Statistics - II (Correlation &amp; Regression) .....</b>	<b>827 – 841</b>
(i)	Curve Fitting .....	827
(ii)	Methods of Least Squares .....	827
(iii)	Correlation .....	831
(iv)	Methods of Estimating Correlation .....	832
(v)	Regression Analysis .....	734
(vi)	Line of Regression of y on x .....	834
(vii)	Line of Regression of x on y .....	835
(viii)	Properties of Regression Coefficients .....	836
(ix)	Angle between Two Lines of Regression .....	836
	<b>Previous Years GATE &amp; ESE Questions .....</b>	<b>840</b>

# Algebra of Matrices

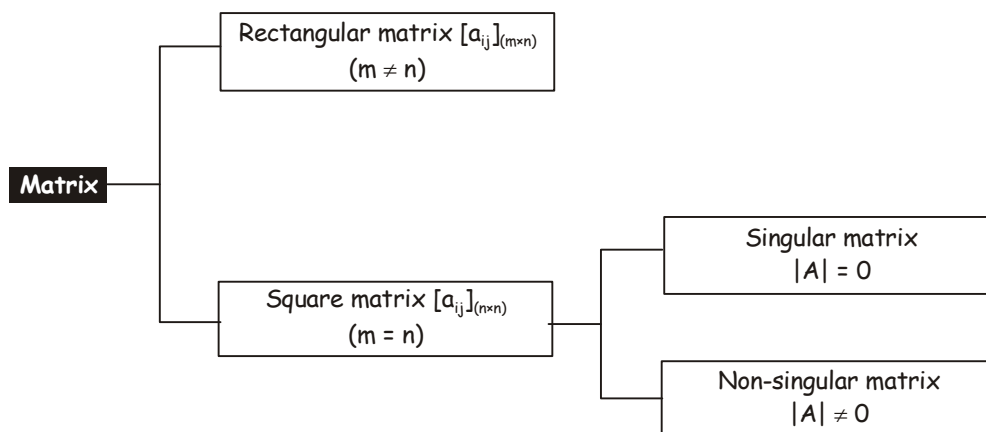
# 1.1

## DEFINITION OF MATRIX

Matrix is a convenient way of storing information in form of  $m$  horizontal rows and  $n$  vertical columns.

Matrix can be represented either  $A = (a_{ij})_{m \times n}$  or  $A = [a_{ij}]_{m \times n}$

## TYPES OF MATRICES



### Rectangular Matrices:

1. **Row Matrix:** A matrix  $A = [a_{ij}]_{m \times n}$  such that  $m = 1$  and  $n > 1$ , matrix is known as row matrix.

e.g.  $A = [a_{ij}]_{1 \times n} = [a_{11}, a_{12}, \dots, a_{1n}]$  or  $[a_1, a_2, \dots, a_n]$  is a row matrix of order  $n$  or matrix of order  $1 \times n$ .

2. **Column Matrix:** A matrix  $A = [a_{ij}]_{m \times n}$  such that  $m > 1$  and  $n = 1$ , matrix is known as row matrix.

e.g.  $A = [a_{ij}]_{n \times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$  or  $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$  is a column matrix of order  $m$  or matrix of order  $m \times 1$ .

3. **Null Matrix or Zero Matrix:** A matrix  $A = [a_{ij}]_{m \times n}$  such that  $[a_{ij}] = 0, \forall i, j \in \mathbb{N}$  is called a zero matrix or Null Matrix i.e.

$$O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. **Horizontal Matrix :** A matrix  $A = [a_{ij}]_{m \times n}$  such that  $m < n$ , matrix is called as horizontal matrix

## 2 Algebra of Matrices

---

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}_{3 \times 4}$$

5. **Vertical Matrix** : A matrix  $A = [a_{ij}]_{m \times n}$  such that  $m > n$ , matrix is called as vertical matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}_{4 \times 3}$$

### Square Matrix:

A matrix in which the number of rows is equal to the number of columns is called a square matrix i.e.  $A = (a_{ij})_{m \times n}$  is a square matrix if and only if  $m = n$ . A matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$
 is a square matrix of order 3. The elements  $a_{11}$ ,  $a_{22}$ ,  $a_{33}$  of the above square matrix are

called its diagonal elements and the diagonal containing these elements is called the principal diagonal or leading diagonal or main diagonal.

**Trace of Matrix:** The sum of the diagonal elements of a square matrix is called **trace** of the matrix.

1. **Diagonal Matrix:** A square matrix is called diagonal matrix if all its non-diagonal elements are zero i.e. in general a matrix  $A = (a_{ij})_{n \times n}$  is called a diagonal matrix if  $a_{ij} = 0$  for  $i \neq j$ ;

For example 
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$
 is a diagonal matrix of order 3'.

2. **Upper Triangular Matrix:** A square matrix  $A = [a_{ij}]$  is called an upper triangular matrix if  $a_{ij} = 0$  whenever  $i > j$ . Thus in an upper triangular matrix all the elements below the principal diagonal are zero. For example,

$$A = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

are  $4 \times 4$  and  $3 \times 3$  upper triangular matrices respectively.

3. **Lower Triangular Matrix** : A square matrix of  $A = [a_{ij}]$  is called a lower triangular matrix if  $a_{ij} = 0$  whenever  $i < j$ . Thus in a lower triangular matrix all the elements above the principal diagonal are zero. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 4 & 2 & 3 & 0 \\ 5 & 4 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$
 are  $4 \times 4$  and  $2 \times 2$  lower triangular matrices respectively.

4. **Scalar Matrix:** If all the elements of a diagonal matrix of order  $n$  are equal, i.e. if  $a_{ii} = k \forall i$ , then the matrix is called a scalar matrix, i.e.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$
 is a scalar matrix of order 3.

5. **Unit or Identity Matrix:** A square matrix is called a unit matrix or identity matrix if all the diagonal elements are unity and non-diagonal elements are zero. e.g.



$$I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

are identity matrices of order  $3 \times 3$  and  $2 \times 2$  respectively.

**6. Sub matrix :** A matrix obtained from a given matrix, say  $A = (a_{ij})_{m \times n}$  by deleting some rows or columns or both

is called a sub matrix of A. For example if  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 1 \\ 7 & 8 & 0 & 2 \\ 1 & 7 & 2 & 3 \end{bmatrix}$  then the matrix  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 7 & 8 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 3 & 5 \\ 8 & 0 \\ 7 & 2 \end{bmatrix}$  are sub matrices

of A.

**7. Equal matrices :**

Two matrices are said to be equal if :

- (i) They are of the same order.
- (ii) The elements in the corresponding positions are equal.

Thus if  $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ . Then  $A = B$

In general if  $A = (a_{ij})_{m \times n}$  and  $B = (b_{ij})_{m \times n}$  are matrices each of order  $m \times n$  and  $a_{ij} = b_{ij}$  for all  $i$  and  $j$  then  $A = B$ .

## PRODUCT OF MATRIX BY A SCALAR (OR CONSTANT)

Let  $A = [a_{ij}]_{m \times n}$  be a matrix of order  $m \times n$  and  $k$  is a constant, then their product is matrix  $kA = [ka_{ij}]_{m \times n}$  i.e. every

element of A is multiplied by  $k$ . For example, if  $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -3 \end{bmatrix}$ . Then we have  $4A = \begin{bmatrix} 8 & -4 & 0 \\ 16 & 20 & -12 \end{bmatrix}$

## ADDITION AND SUBTRACTION OF MATRICES

Let A and B be two matrices of the same order, then their sum  $A + B$  is defined as the matrix each element of which is the sum of the corresponding elements of A and B.

In general if  $A = (a_{ij})_{m \times n}$  and  $B = (b_{ij})_{m \times n}$  then their sum is defined by the matrix.

$$C = A + B = (C_{ij})_{m \times n}$$

where  $c_{ij} = a_{ij} + b_{ij}$  and  $i = 1, 2, \dots, m$  and  $j = 1, 2, 3, \dots, n$

$$\text{If } A = \begin{bmatrix} 4 & 2 & 5 \\ 11 & 13 & -6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\text{Then } A + B = \begin{bmatrix} 4+1 & 2+0 & 5+2 \\ 11+3 & 13+1 & -6+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 14 & 14 & -2 \end{bmatrix}$$

Similarly, if A and B are two matrices of the same order, then their difference is defined by

$$A - B = \begin{bmatrix} 4-1 & 2-0 & 5-2 \\ 11-3 & 13-1 & -6-4 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 8 & 12 & -10 \end{bmatrix}$$

### Properties of Matrix Addition

- (i) Matrix addition is commutative :