

ENGINEERING MATHEMATICS

(For ESE & GATE Exam)

(CE, ME, PI, CH, EC, EE, IN, CS, IT)

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- 280 topics under 32 chapters in 8 units
- 640 Solved Examples for comprehensive understanding
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- Only book having complete theory on ESE & GATE Pattern
- Comprising conceptual questions marked with “*” for quick revision



Office : F-126, (Lower Basement), Katwaria Sarai, New Delhi-110016 • **Phone :** 011-26522064
Mobile : 8130909220, 9711853908 • **E-mail:** info.publications@iesmaster.org, info@iesmaster.org
Web : iesmasterpublications.com, iesmaster.org



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F-126, (Lower Basement), Katwaria Sarai, New Delhi-110016

Phone : 011-26522064, **Mobile :** 8130909220, 9711853908

E-mail : info.publications@iesmaster.org

Web : iesmasterpublications.com

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PREFACE

Science and technology has contributed greatly to the progress of the human race, and thus has made our life quite easier and comfortable. Engineers have played a vital role in the advancements of the technological world. However, with advancements in technology, there come challenges. It is here that the role of Engineering Mathematics comes in play to address the technological challenges posed by the modern era.

Engineering Mathematics is the art of applying mathematics to complicated real-world problems. It is a mix of mathematical theory, practical engineering and scientific computing.

Keeping in mind the importance of Engineering Mathematics skill in modern engineers, in the year 2017, the Union Public Service Commission (UPSC) introduced Engineering Mathematics as a common paper in the syllabus for Engineering Services Examination (ESE), and technical paper for Electrical (EE) stream. It has already been given a weightage of 15% in the Graduate Aptitude Test in Engineering (GATE).

With an objective to develop an ESE or GATE aspirant's numerical abilities and calculation skills, IES Master has come up with this Engineering Mathematics book that brings them face to face with **280 topics under 32 chapters in 8 units**, along with previous years questions from **GATE (last 32 years), and ESE (last 7 years)** and their detailed solutions. Equipped with all this, students can easily decide, how much time to allocate on each chapter based on the number of questions asked in that particular exam.

It is the only book in the market that includes complete theory exactly on the basis of ESE & GATE exams pattern. After each topic, the book includes more than **640 solved examples** for concept building & easy learning. To save students time during revision, all the previously asked conceptual questions are marked with '*' symbol.

Our special thanks to the entire IES Master team for their continuous support in bringing out the book. We strongly believe that this book will help students in their journey to success. Suggestions from students, teachers & educators for further improvement in the book are always welcome.

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Algebra of Matrices

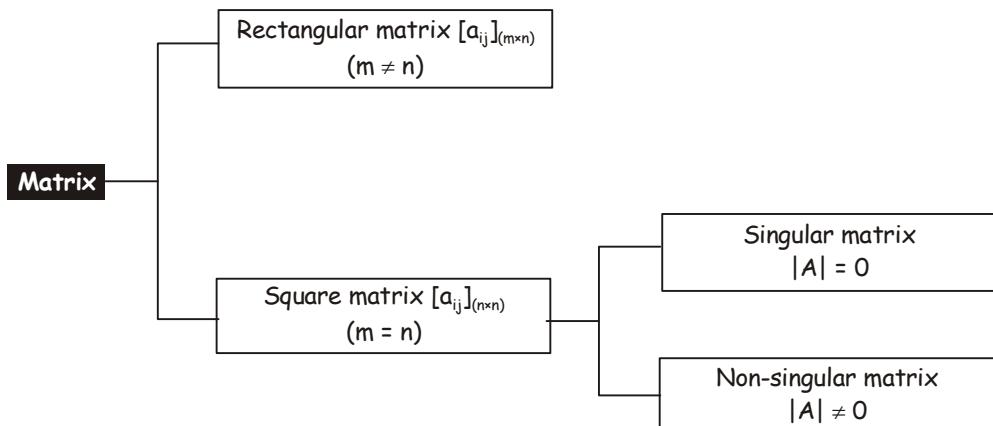
1.1

DEFINITION OF MATRIX

Matrix is a convenient way of storing information in form of m horizontal rows and n vertical columns.

Matrix can be represented either $A = (a_{ij})_{m \times n}$ or $A = [a_{ij}]_{m \times n}$

TYPES OF MATRICES



Rectangular Matrices:

1. **Row Matrix:** A matrix $A = [a_{ij}]_{m \times n}$ such that $m = 1$ and $n > 1$, matrix is known as row matrix.

e.g. $A = [a_{ij}]_{1 \times n} = [a_{11}, a_{12}, \dots, a_{1n}]$ or $[a_1, a_2, \dots, a_n]$ is a row matrix of order n or matrix of order $1 \times n$.

2. **Column Matrix:** A matrix $A = [a_{ij}]_{m \times n}$ such that $m > 1$ and $n = 1$, matrix is known as column matrix.

e.g. $A = [a_{ij}]_{n \times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ or $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ is a column matrix of order m or matrix of order $m \times 1$.

3. **Null Matrix or Zero Matrix:** A matrix $A = [a_{ij}]_{m \times n}$ such that $[a_{ij}] = 0, \forall i, j \in N$ is called a zero matrix or Null Matrix i.e.

$$O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. **Horizontal Matrix :** A matrix $A = [a_{ij}]_{m \times n}$ such that $m < n$, matrix is called as horizontal matrix

2 Algebra of Matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}_{3 \times 4}$$

5. **Vertical Matrix** : A matrix $A = [a_{ij}]_{m \times n}$ such that $m > n$, matrix is called as vertical matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}_{4 \times 3}$$

Square Matrix:

A matrix in which the number of rows is equal to the number of columns is called a square matrix i.e. $A = (a_{ij})_{m \times n}$ is a square matrix if and only if $m = n$. A matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$

is a square matrix of order 3. The elements a_{11}, a_{22}, a_{33} of the above square matrix are

called its diagonal elements and the diagonal containing these elements is called the principal diagonal or leading diagonal or main diagonal.

Trace of Matrix: The sum of the diagonal elements of a square matrix is called **trace** of the matrix.

1. **Diagonal Matrix:** A square matrix is called diagonal matrix if all its non-diagonal elements are zero i.e. in general a matrix $A = (a_{ij})_{n \times n}$ is called a diagonal matrix if $a_{ij} = 0$ for $i \neq j$;

For example $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix of order 3'.

2. **Upper Triangular Matrix:** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ whenever $i > j$. Thus in an upper triangular matrix all the elements below the principal diagonal are zero. For example,

$$A = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

are 4×4 and 3×3 upper triangular matrices respectively.

3. **Lower Triangular Matrix :** A square matrix of $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0$ whenever $i < j$. Thus in a lower triangular matrix all the elements above the principal diagonal are zero. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 4 & 2 & 3 & 0 \\ 5 & 4 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

are 4×4 and 2×2 lower triangular matrices respectively.

4. **Scalar Matrix:** If all the elements of a diagonal matrix of order n are equal, i.e. if $a_{ii} = k \forall i$, then the matrix is called a scalar matrix, i.e.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is a scalar matrix of order 3.

5. **Unit or Identity Matrix:** A square matrix is called a unit matrix or identity matrix if all the diagonal elements are unity and non-diagonal elements are zero. e.g.

$$I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

are identity matrices of order 3×3 and 2×2 respectively.

6. Sub matrix : A matrix obtained from a given matrix, say $A = (a_{ij})_{m \times n}$ by deleting some rows or columns or both

is called a sub matrix of A . For example if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 1 \\ 7 & 8 & 0 & 2 \\ 1 & 7 & 2 & 3 \end{bmatrix}$ then the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 7 & 8 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 5 \\ 8 & 0 \\ 7 & 2 \end{bmatrix}$ are sub matrices of A .

7. Equal matrices :

Two matrices are said to be equal if :

- (i) They are of the same order.
- (ii) The elements in the corresponding positions are equal.

Thus if $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Then $A = B$

In general if $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ are matrices each of order $m \times n$ and $a_{ij} = b_{ij}$ for all i and j then $A = B$.

PRODUCT OF MATRIX BY A SCALAR (OR CONSTANT)

Let $A = [a_{ij}]_{m \times n}$ be a matrix of order $m \times n$ and k is a constant, then their product is matrix $kA = [ka_{ij}]_{m \times n}$ i.e. every element of A is multiplied by k . For example, if $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -3 \end{bmatrix}$. Then we have $4A = \begin{bmatrix} 8 & -4 & 0 \\ 16 & 20 & -12 \end{bmatrix}$

ADDITION AND SUBTRACTION OF MATRICES

Let A and B be two matrices of the same order, then their sum $A + B$ is defined as the matrix each element of which is the sum of the corresponding elements of A and B .

In general if $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ then their sum is defined by the matrix.

$$C = A + B = (C_{ij})_{m \times n}$$

where $C_{ij} = a_{ij} + b_{ij}$ and $i = 1, 2, \dots, m$ and $j = 1, 2, 3, \dots, n$

If $A = \begin{bmatrix} 4 & 2 & 5 \\ 11 & 13 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$

$$\text{Then } A + B = \begin{bmatrix} 4+1 & 2+0 & 5+2 \\ 11+3 & 13+1 & -6+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 14 & 14 & -2 \end{bmatrix}$$

Similarly, if A and B are two matrices of the same order, then their difference is defined by

$$A - B = \begin{bmatrix} 4-1 & 2-0 & 5-2 \\ 11-3 & 13-1 & -6-4 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 8 & 12 & -10 \end{bmatrix}$$

Properties of Matrix Addition

- (i) Matrix addition is commutative :

4 Algebra of Matrices

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be matrices of the same order $m \times n$ then $A + B = B + A$.

(ii) Matrix addition is associative :

Let A, B, C can be the matrices of the same order, Then $(A + B) + C = A + (B + C)$.

(iii) Cancellation law for matrix addition :

Let A, B, C be the matrices of the same order, then

$A + B = A + C$ holds if and only if $B = C$.

MULTIPLICATION OF MATRICES

The product AB of two matrices A and B is possible only when the number of columns in A is equal to the number of rows in B. Such matrices are said to be conformable for multiplication.

Let $A = [a_{ij}]_{m \times n}$ be a matrix of order $m \times n$ and $B = [b_{jk}]_{n \times p}$ be a matrix of order $n \times p$, then the product AB is defined as a matrix $C = [c_{ik}]_{m \times p}$ of order $m \times p$

where

$$c_{ik} = \sum_{j=1}^m a_{ij}b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk}$$

i.e. $(i, k)^{\text{th}}$ element of AB = sum of the products of the elements of i^{th} row of A with the corresponding elements of k^{th} column of B, i.e.

Example 1 : If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$. Show that $AB \neq BA$

Solution : $AB = \begin{bmatrix} 1.2 - 2.4 + 3.2 & 1.3 - 2.5 + 3.1 \\ -4.2 + 2.4 + 5.2 & -4.3 + 2.5 + 5.1 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}_{2 \times 2}$

Since number of columns in B is equal to number of rows in A, so BA is also defined.

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 27 \\ -16 & 2 & 27 \\ -2 & -2 & 11 \end{bmatrix}$$

Hence $AB \neq BA$.

Thus matrix multiplication is not commutative in general.

Example 2 : If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 1 \end{bmatrix}$. Find AB. Will BA exist?

Solution : $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 10 & 1 \\ 15 & 5 \end{bmatrix}$

Now, since number of columns of B is 2 and number of rows in A are 3. So BA does not exist. Therefore if AB is defined, it is not at all necessary that the product BA is also defined.

Properties of Matrix Multiplication

1. Multiplication of matrices is not commutative i.e. $AB \neq BA$

2. Multiplication of matrices is associative, i.e. $A(BC) = (AB)C$
3. Matrix multiplication is distributive with respect to addition i.e. $A(B + C) = AB + AC$
where A, B, C are any three matrices of order $m \times n$, $n \times p$, $n \times p$ respectively.
4. **Positive integral power of square matrix :**

The product of AA is defined only when A is square matrix of order n. We shall denote it as A^2 . If m and n are any positive integers, then we have $A^m A^n = A^{m+n}$

5. **Zero Divisor :**

$AB = O$ does not necessarily imply that at least one of the matrices A and B must be zero matrix i.e. the product of two matrices can be zero matrix while neither of them is a zero matrix. Such matrices are said to be zero divisor. For example; if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \text{ then } AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e. neither the matrix A nor matrix B is a zero matrix but their product matrix AB is zero matrix.

6. **Multiplication with Identity Matrix :**

If A be $n \times n$ matrix and I_n is a unit matrix of order n, then $AI_n = I_n A = A$

i.e. a matrix remains unaltered when it is multiplied by a unit matrix of same order.

MINORS OF MATRIX

The determinant value of the square matrix obtained from the original matrix of any order by the omission of the rows and columns is called a minor of a matrix. For example

If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 7 & 1 \\ 8 & 2 & 7 & 3 \end{bmatrix}_{3 \times 4}$ is a matrix of order 3×4 . Then minors of A are $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 7 \end{vmatrix}$; $\begin{vmatrix} 0 & 3 \\ 8 & 2 \end{vmatrix}$ & $\begin{vmatrix} 7 & 1 \\ 8 & 2 \end{vmatrix}$ etc.

COFACTORS OF A MATRIX

The cofactors of the element a_{ij} is defined as

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

where $|M_{ij}|$ is the determinant obtained by deleting the i^{th} row and j^{th} column from given matrix.

EXPANSION BY COFACTORS

We have row expansion of a determinant of a general 3×3 matrix, that is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Since $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$, $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = M_{12}$, and $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = M_{13}$. So we have,

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Since by the definition of cofactors, we have

6 Algebra of Matrices

$$A_{11} = (-1)^{1+1} M_{11} = M_{11},$$

$$A_{12} = (-1)^{1+2} M_{12} = -M_{12},$$

and

$$A_{13} = (-1)^{1+3} M_{13} = M_{13}.$$

PROPERTIES OF DETERMINANTS

1. $|A'| = |A|$
2. If we interchange any two rows or columns then sign of determinants change.
3. If any 2 rows or any 2 columns in a determinant are identical (or proportional) then the value of the determinant is zero.
4. Multiplying a determinant by K means multiplying the elements of only one row (or one column) by K.

e.g.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

then

$$2|A| = 2 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = 8 - 12 = -4 = 2(-2)$$

5. If elements of a row in a determinant can be expressed as the sum of two or more elements then the given determinant can be expressed as the sum of 2 or more determinants.

e.g.

$$|A| = \begin{vmatrix} a+c & b+d \\ e & f \end{vmatrix} = \begin{vmatrix} a & b \\ e & f \end{vmatrix} + \begin{vmatrix} c & d \\ e & f \end{vmatrix}$$

6. If we apply the operations like $R_i \rightarrow R_i + KR_j$ or $C_i \rightarrow C_i + KC_j$, the value of the determinant remains unchanged.
7. Thus, the determinant is the sum of products of elements of any row and their corresponding cofactors. i.e.,

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

where i represents rows. This expansion is called the expansion by cofactors.

Similarly, we can, expand the determinant as the sum of the product of elements of any column and the corresponding cofactors of the elements of the same column. That is,

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}$$

8. This expansion has a very important property that sum of products of elements of the i^{th} row and the corresponding cofactors of the elements of the i_1^{th} row is zero, that is,

$$a_{i1}A_{i_11} + a_{i2}A_{i_12} + a_{i3}A_{i_13} = 0, \text{ in the case of the determinant of general } 3 \times 3 \text{ matrix where } i \neq i_1. \text{ Thus, if we choose}$$

$$i = 1, i_1 = 3$$

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = a_{11} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{13} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}(a_{12}a_{23} - a_{13}a_{22}) - a_{12}(a_{11}a_{23} - a_{13}a_{21}) + a_{13}(a_{11}a_{22} - a_{12}a_{21}) \\ &= a_{11}a_{12}a_{23} - a_{11}a_{13}a_{22} - a_{12}a_{11}a_{23} + a_{12}a_{13}a_{21} + a_{13}(a_{11}a_{22}) - a_{13}a_{12}a_{21} = 0 \end{aligned}$$

The same result holds good for the column expansion. That is $a_{1j}A_{1j_1} + a_{2j}A_{2j_1} + a_{3j}A_{3j_1} = 0$ where $j \neq j_1$

ADJOINT OF MATRIX

Let $A = [a_{ij}]_{n \times n}$ be a square matrix and A_{ij} is the cofactor of a_{ij} . Then the transpose of the matrix B of cofactors of the elements of A is known as the adjoint of A and is denoted by $\text{Adj } A = B'$ where $B = A_{ij}$ = cofactor matrix of A.